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Mapping Noether Identities into Bianchi Identities in General Relativistic Theories of Gravity and in the Field Theory of Static Lattice Defects

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Noether identities resulting from external symmetries represent "conservation" laws in relativistic field theories and balance laws in 3-dimensional continuum statics, respectively. In a suitably selected 4-dimensional non-Euclidean spacetime (3-dimensional stress space), the momentum currents (stresses) entering the conservation (balance) laws can be mapped such that the Noether identities become Bianchi identities, or irreducible pieces thereof. Using a metric-affine space with independent metric $g_{\alpha\beta}$ and connection $\Gamma_\alpha^{\ \beta}$, we derive the following types of mapping prescriptions: momentum current → (contraction of) curvature; spin current → torsion; shear current → trace-free nonmetricity; dilation current → Weyl 1-form. The last two mappings constitute the main result. The mapping of the dilation current turns out to be exceptional, since it does not yield a nontrivial Bianchi identity.

1. INTRODUCTION

In general relativity theory we have the energy-momentum law[5]

$$\overset{\circ}{D}\sigma_\alpha = 0 \tag{1}$$

This paper is dedicated to Prof. Ekkehart Kröner, Stuttgart, on the occasion of his 70th birthday.

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[5]Our notation is as follows: vector basis $\{e_\alpha\}$, dual basis of 1-forms $\{\vartheta^\alpha\}$, i.e., $e_\alpha \rfloor \vartheta^\beta = \delta_\alpha^\beta$, $\alpha, \beta, \ldots = 0, 1, 2, 3$ (or 1, 2, 3 in continuum mechanics). The metric reads $g = g_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$; in an orthonormal basis we have $g = o_{\alpha\beta}\vartheta^{(\mathrm{orth})\alpha} \otimes \vartheta^{(\mathrm{orth})\beta}$ with $o_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. Exterior product $\wedge$, interior product (of a vector with a form) $\rfloor$, exterior derivative d, exterior covariant derivative D (or $\overset{\circ}{D}$ in Riemannian space), $*$ means the Hodge dual, $\eta = |\det g_{\mu\nu}|^{1/2}\vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3$ (in continuum mechanics omit ϑ^0) is the volume form, $\eta_\alpha = {}^*\vartheta_\alpha = e_\alpha \rfloor \eta$, $\eta_{\alpha\beta} = {}^*(\vartheta_\alpha \wedge \vartheta_\beta) = e_\beta \rfloor \eta_\alpha$, etc.

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with σ_α as (the 3-form of the symmetric) energy-momentum current and $\overset{\circ}{D}$ as the covariant exterior derivative corresponding to the Riemannian connection. This law is known to represent a *Noether* identity[6] resulting from the invariance of the matter Lagrangian under diffeomorphisms of the underlying Riemannian spacetime.

The energy-momentum current σ_α is the source in Einstein's field equation (Kopczyński, 1987; Mielke, 1987; Thirring, 1986; Trautman, 1973, 1984; Wallner, 1982):

$$\tfrac{1}{2}\overset{\circ}{R}{}^{\beta\gamma} \wedge \eta_{\beta\gamma\alpha} = \kappa\sigma_\alpha \tag{2}$$

where $\overset{\circ}{R}{}^{\alpha\beta}$ is the Riemann curvature 2-form, $\eta_{\alpha\beta\gamma} = {}^*(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma)$, and κ is the gravitational constant. For $\overset{\circ}{R}{}^{\beta\gamma}$ the Bianchi identity $\overset{\circ}{D}\overset{\circ}{R}{}^{\beta\gamma} = 0$ is known to hold and, in particular, its contraction

$$\overset{\circ}{D}(\overset{\circ}{R}{}^{\beta\gamma} \wedge \eta_{\beta\gamma\alpha}) \equiv 0 \tag{3}$$

Equation (3) represents a geometrical identity which is valid in any Riemannian spacetime.

If we substitute the Einstein equation (2) into the identity (3), then we find the energy-momentum law (1). Accordingly, provided the field equation (2) is fulfilled, the energy-momentum law is always an automatic by-product of the Bianchi identity. In this sense we have mapped the Noether identity (1), via the field equation (2), into the contracted Bianchi identity (3).

We have shown (Hehl and McCrea, 1986) that this idea of an automatic "conservation" of energy-momentum can be extended to the *angular momentum* law if we turn to the Einstein–Cartan–Sciama–Kibble theory of gravity (ECT) with its Riemann–Cartan spacetime. In the present article we prove that this concept can be still further generalized to the (trace-free part of the) so-called *hypermomentum* law (Hehl *et al.*, 1989), if we extend the geometry of spacetime to a metric-affine one and use a modified Hilbert–Einstein type of gravitational Lagrangian.

The intuitive idea of the last-mentioned extension of the "automatic conservation" of currents emerged during a recent lecture of Kröner (1990) in which the field theory of crystal dislocations (Kröner, 1981) within the realm of *3-dimensional* continuum statics was enriched by the new concept of continuously distributed point stacking faults. Whereas from the point of view of continuum statics, dislocations yield (2-forms of) spin moment stress $\tau_{\alpha\beta} = -\tau_{\beta\alpha}$ (intrinsic double stress with moment (Kröner, 1963*a,b*; Hehl and Kröner, 1965), the point stacking faults are expected to induce, in the 3-dimensional body under consideration, (2-forms of) intrinsic double

[6]Since (1) is valid only provided the material Euler–Lagrange equations have been satisfied, we should, strictly speaking, use the terminology "weak identity" or "identity on shell." However, we shall refer to it simply as an identity.

stress *without* moment, $(\Delta_{\alpha\beta}+\Delta_{\beta\alpha})/2\equiv\Delta_{(\alpha\beta)}$. Therefore, on the stress side, we find, within the framework of the field theory of lattice defects, (force) stress Σ_α and intrinsic *hyperstress* $\Delta_{\alpha\beta}=\tau_{\alpha\beta}+\Delta_{(\alpha\beta)}$ with

$$\tau_{\alpha\beta}:=(\Delta_{\alpha\beta}-\Delta_{\beta\alpha})/2\equiv\Delta_{[\alpha\beta]}$$

In the absence of body forces and intrinsic body double forces, the linearized equilibrium conditions for such a continuum with microstructure read in Cartesian coordinates[7] [see Mindlin (1964): put $\Sigma=\tau_{\mathrm{Mind}}+\sigma_{\mathrm{Mind}}$, $\sigma=\tau_{\mathrm{Mind}}$, $\Delta=\mu_{\mathrm{Mind}}$]

$$d\Sigma_\alpha=0 \tag{4}$$

$$d\Delta_{\alpha\beta}+\vartheta_\alpha\wedge\Sigma_\beta-\sigma_{\alpha\beta}=0 \tag{5}$$

where $\sigma_{\alpha\beta}=\sigma_{\beta\alpha}:=\vartheta_\alpha\wedge\sigma_\beta$ represents the symmetric Cauchy stress. Clearly, for vanishing hyperstress, we recover the equilibrium conditions $d\sigma_\alpha=0$ and $\vartheta_{[\alpha}\wedge\sigma_{\beta]}=0$ of classical continuum statics.

For the classical continuum, Cartan (1922, 1986) and Schaefer (1953) introduced a 3-dimensional Riemannian *stress space* with curvature $\mathring{R}^{\alpha\beta}$. They showed that, by the identification prescription $\sigma_\alpha=(1/2K)\mathring{R}^{\beta\gamma}\wedge\eta_{\beta\gamma\alpha}$, where K is a constant with the dimension $(\text{force})^{-1}$, the equilibrium conditions are automatically satisfied by virtue of the Bianchi identity. In the continuum theory of dislocations it is necessary to include additionally spin moment stress $\tau_{\alpha\beta}$. Amari (1981), Kröner (1963*a*,*b*), Minagawa (1962), and Stojanovic (1963) have pointed out that under those circumstances one must resort to a Riemann-*Cartan* stress space with torsion (compare Cartan 1922, 1986). If T^α denotes its *torsion* 2-form, they found that the identifications

$$\Sigma_\alpha=\frac{1}{2K}R^{\gamma\delta}\wedge\eta_{\gamma\delta\alpha} \tag{6}$$

$$\tau_{\alpha\beta}=\frac{1}{2K}T^{\gamma}\wedge\eta_{\gamma\alpha\beta} \tag{7}$$

(Hehl and McCrea, 1986, Table IV) translate the equilibrium condition (4) and the *antisymmetric part* of (5) into the 2nd and 1st Bianchi identities,

$$DR_\alpha{}^\beta\equiv 0 \tag{8}$$

and

$$DT^\alpha-\vartheta^\beta\wedge R_\beta{}^\alpha\equiv 0 \tag{9}$$

respectively.[8]

[7]From reference Kröner (1987) it is clear that the mapping procedures formulated below also should be valid if full nonlinearity is admitted as well as arbitrary coordinates.

[8]In 3 dimensions, the Bianchi identities need not be contracted for that purpose.

If, in addition to spin moment stress, we now introduce intrinsic hyperstress without moment $\Delta_{(\alpha\beta)}$ and the corresponding equilibrium condition, namely the symmetric part of (5), the next step is not too far-fetched. In a Riemann–Cartan space the *nonmetricity* 1-form

$$Q_{\alpha\beta} := -Dg_{\alpha\beta} \tag{10}$$

vanishes, where $g_{\alpha\beta}$ denotes the components of the metric g with respect to an arbitrary frame. If we allow for nonmetricity,[9] thereby widening the Riemann–Cartan space to a metric-affine space with arbitrary metric g and arbitrary connection 1-form $\Gamma_\alpha^{\ \beta}$, then the zeroth Bianchi identity reads (Schouten, 1954; Hehl *et al.*, 1989)

$$DQ_{\alpha\beta} \equiv 2R_{(\alpha\beta)} \tag{11}$$

Accordingly, the ansatz

$$\Delta_{(\alpha\beta)} \sim \frac{1}{2K} Q_{\gamma(\alpha} \wedge \eta^{\gamma}{}_{\beta)} \tag{12}$$

should be a likely candidate for the mapping into the symmetric part of (5).

Apart from a factor, equation (12) seems to be the only possibility, since $\Delta_{(\alpha\beta)}$ is an $(n-1)$-form, $Q_{\alpha\beta}$ a 1-form, and $\eta^{\gamma}{}_{\beta}$ an $(n-2)$-form, if n is the dimension of the space. However, the trace of the right-hand side of (12) vanishes. In other words, equation (12) involves only the trace-free part of $\Delta_{(\alpha\beta)}$. Therefore, equation (12) should rather read

$$\not{\Delta}_{\alpha\beta} := \Delta_{(\alpha\beta)} - \frac{1}{n} g_{\alpha\beta} \Delta_{\gamma}^{\ \gamma} = \frac{1}{2K} Q_{\gamma(\alpha} \wedge \eta^{\gamma}{}_{\beta)} \tag{13}$$

We shall see below in detail that this is indeed the correct ansatz.

This means that our mapping procedures (6), (7),[10] together with (13), only take care of the equilibrium conditions (4) and the trace-free part of (5), whereas the *trace*, and only the trace, of (5), i.e.,

$$d\Delta_{\gamma}^{\ \gamma} + \vartheta^{\gamma} \wedge \Sigma_{\gamma} - \sigma_{\gamma}^{\ \gamma} = 0 \tag{14}$$

is left without geometrical image. A remedy for that could be an identification (Hehl *et al.*, 1981) of the type

$$\Delta_{\gamma}^{\ \gamma} \sim \frac{1}{K} {}^*Q_{\gamma}^{\ \gamma} \tag{15}$$

linking the trace $\Delta := \Delta_{\gamma}^{\ \gamma}$ of the intrinsic hyperstress with the Weyl 1-form

[9]The idea of introducing nonmetricity in continuum physics was suggested by Günther and Zorawski and by Kröner (Günther and Zorawski, 1985, and Zorawski and Günther, unpublished; Kröner, 1981).

[10]Equation (7) picks up a $Q_{\alpha\beta}$-dependent piece in an (L_n, g); see Section 5, (M2).

$Q := (1/n)Q^{\gamma}_{\gamma}$. We will find out below, however, that this does not lead to the desired geometrical image.

Up to now we have used heuristic arguments. In the following, however, we present a mathematically consistent framework for a toy model of a gravitational gauge theory with a modified Hilbert–Einstein-type Lagrangian in a *metric-affine* space. In Sections 2 and 3 we deal with a general n-dimensional space. In subsequent sections, although much of what we do is applicable to n dimensions, we explicitly *derive* the results for 4-dimensional spacetime and the analogous results for the 3-dimensional space of classical continuum mechanics are simply exhibited.

In Section 2, we look into the Bianchi identities for an n-dimensional metric-affine space (L_n, g) and into their projective invariance. The Noether identities of a matter field minimally coupled to the (L_n, g) will be spelled out in Section 3. In Section 4 we present in tabular form the results that we intend to derive in the following sections. It was felt that by placing the summary of our results before their actual derivation, our motivation would become more transparent. In Section 5 we write down the Lagrangian, which we use as a tool for getting hold of the mapping prescriptions, and derive the corresponding field equations. It is these field equations that will provide the required mappings between the Noether and Bianchi identities. As one would expect from Hehl and McCrea (1986), the mappings will involve relations between *irreducible pieces* of Noether and *irreducible pieces* of Bianchi. Since the Noether identities are n-forms in an n-dimensional space, their irreducible decomposition is a trivial matter. This is not so for Bianchi and Section 6 is devoted to the derivation of the irreducible parts of the 0th, 1st, and 2nd Bianchi identities in a metric-affine space. The way is then clear for the derivation of the Noether–Bianchi mappings in Section 7. Finally, in Section 8, we summarize our results.

2. BIANCHI IDENTITIES IN A METRIC-AFFINE SPACE (L_n, g) AND THE VOLUME-PRESERVING CONNECTION

The basic variables, that is, the potentials, of a metric-affine space (L_n, g) are the metric $g = g_{\alpha\beta}\vartheta^{\alpha} \otimes \vartheta^{\beta}$, the 1-form basis ϑ^{α}, and the connection 1-form $\Gamma_{\alpha}{}^{\beta}$. Gauge-theoretically, the metric g is a subsidiary tensor field for the onset of the group reduction from the $GL(n, R)$ down to the $SO(1, n-1)$. Picking an orthonormal basis $\overset{\text{orth}}{\vartheta}{}^{\alpha}$ would allow us to transform the metric coefficients $g_{\alpha\beta}$ to their (constant) Minkowskian values $o_{\alpha\beta} = \operatorname{diag}(-1, 1, \ldots, 1)$ at each point. However, we will not introduce such a restrictive gauge yet. Rather, we treat the $g_{\alpha\beta}$ as formally independent variables.

The field strengths corresponding to these potentials are the non-metricity 1-form

$$Q_{\alpha\beta} := -Dg_{\alpha\beta} \tag{16}$$

the torsion 2-form

$$T^{\alpha} := D\vartheta^{\alpha} \tag{17}$$

and the curvature 2-form

$$R_{\alpha}{}^{\beta} := d\Gamma_{\alpha}{}^{\beta} + \Gamma_{\gamma}{}^{\beta} \wedge \Gamma_{\alpha}{}^{\gamma} \tag{18}$$

By exterior differentiation we find the zeroth, first, and second Bianchi identities,

$$DQ_{\alpha\beta} \equiv 2R_{(\alpha\beta)} \tag{19}$$

$$DT^{\alpha} \equiv \vartheta^{\beta} \wedge R_{\beta}{}^{\alpha} \tag{20}$$

$$DR_{\alpha}{}^{\beta} \equiv 0 \tag{21}$$

Let $A_{\alpha}{}^{\beta}$ be a $GL(n, R)$ tensor-valued 1-form. Then, by the deformation

$$'\Gamma_{\alpha}{}^{\beta} = \Gamma_{\alpha}{}^{\beta} - A_{\alpha}{}^{\beta} \tag{22}$$

the old connection $\Gamma_{\alpha}{}^{\beta}$ is carried over into a new connection $'\Gamma_{\alpha}{}^{\beta}$. Since the Bianchi identities (19)-(21) are valid for any connection, they can also be formulated in terms of the dashed connection.[11]

Motivated by our ansatz (13), we expect a connection ${}^{\dagger}\Gamma_{\alpha}{}^{\beta}$ with vanishing Weyl 1-form to enter in an essential way into our considerations. This *volume-preserving connection* (Hehl *et al.*, 1981, 1988) results from a specific deformation of $\Gamma_{\alpha}{}^{\beta}$ with $A_{\alpha}{}^{\beta} = \frac{1}{2}Q\delta_{\alpha}^{\beta}$, namely

$${}^{\dagger}\Gamma_{\alpha}{}^{\beta} := \Gamma_{\alpha}{}^{\beta} - \tfrac{1}{2}Q\delta_{\alpha}^{\beta} \tag{23}$$

where the *Weyl* 1-form is defined according to (Schouten, 1954)

$$Q := \frac{1}{n} g^{\alpha\beta} Q_{\alpha\beta} = \frac{1}{n} Q_{\gamma}^{\gamma} = \frac{2}{n} \Gamma_{\gamma}^{\gamma} - \frac{1}{n} g^{\alpha\beta}\, dg_{\alpha\beta} \tag{24}$$

It has the property that

$${}^{\dagger}D\eta_{\alpha\beta\gamma\delta} = 0 \tag{25}$$

where $\eta_{\alpha\beta\gamma\delta} = e_{\delta} \rfloor e_{\gamma} \rfloor e_{\beta} \rfloor e_{\alpha} \rfloor \eta$ are the components of the volume 4-form η (the generalization to n dimensions is obvious), and hence

$$d|\det g_{\alpha\beta}| - {}^{\dagger}\Gamma_{\gamma}^{\gamma}|\det g_{\alpha\beta}| = 0 \tag{26}$$

[11]If $A_{\alpha}{}^{\beta}$ is chosen to be the non-Riemannian part of the connection $\Gamma_{\alpha}{}^{\beta}$, then $T_{\alpha}{}^{\beta} = \mathring{\Gamma}_{\alpha}{}^{\beta}$ is the Levi-Civita connection and the Bianchi identities boil down to those of a Riemannian space.

For the Weyl 1-form of ${}^{\dagger}\Gamma_{\alpha}{}^{\beta}$ we find indeed

$$ {}^{\dagger}Q := \frac{1}{n}\, g^{\alpha\beta\,\dagger}Q_{\alpha\beta} = -\frac{1}{n}\, g^{\alpha\beta\,\dagger}Dg_{\alpha\beta} = 0 \tag{27} $$

since ${}^{\dagger}Dg_{\alpha\beta} = Dg_{\alpha\beta} + Qg_{\alpha\beta}$.

Using (23) and the definitions (16)-(18), we find straightforwardly

$$ {}^{\dagger}Q_{\alpha\beta} = Q_{\alpha\beta} - Qg_{\alpha\beta} \tag{28} $$

$$ {}^{\dagger}T^{\alpha} = T^{\alpha} - \tfrac{1}{2}Q \wedge \vartheta^{\alpha} \tag{29} $$

$$ {}^{\dagger}R_{\alpha}{}^{\beta} = R_{\alpha}{}^{\beta} - \tfrac{1}{2}\, dQ\, \delta_{\alpha}^{\beta} \tag{30} $$

Clearly, (27) is a special case of (28). As already mentioned, the Bianchi identities keep their respective shapes,

$$ {}^{\dagger}D\, {}^{\dagger}Q_{\alpha\beta} \equiv 2\, {}^{\dagger}R_{(\alpha\beta)} \tag{31} $$

$$ {}^{\dagger}D\, {}^{\dagger}T^{\alpha} \equiv \vartheta^{\beta} \wedge {}^{\dagger}R_{\beta}{}^{\alpha} \tag{32} $$

$$ {}^{\dagger}D\, {}^{\dagger}R_{\alpha}{}^{\beta} \equiv 0 \tag{33} $$

3. NOETHER IDENTITIES FOR A MINIMALLY COUPLED MATTER FIELD IN AN (L_n, g)

Starting from a special-relativistic first-order Lagrangian, the matter field has now to be immersed in the (L_n, g) geometry by minimal coupling in order to end up with a $GL(n, R)$-invariant Lagrangian:

$$ L^{\mathrm{SR}}(o_{\alpha\beta}, \Psi, d\Psi) \to L(g_{\alpha\beta}, \vartheta^{\alpha}, \Psi, D\Psi) \tag{34} $$

The coframe ϑ^{α} no longer needs to be orthonormal, i.e., the metric $g_{\alpha\beta}$ and the coframe ϑ^{α}, besides the connection $\Gamma_{\alpha}{}^{\beta}$ and the matter field $\Psi(x)$, are regarded as formally independent variables.

If we vary the Lagrangian n-form with respect to the potentials, we find the following *matter currents*:

$$ \sigma^{\alpha\beta} := 2\frac{\delta L}{\delta g_{\alpha\beta}} = 2\frac{\partial L}{\partial g_{\alpha\beta}} \qquad \text{metric stress-energy} \tag{35} $$

$$ \Sigma_{\alpha} := \frac{\delta L}{\delta \vartheta^{\alpha}} = \frac{\partial L}{\partial \vartheta^{\alpha}} \qquad \text{canonical energy-momentum} \tag{36} $$

$$ \Delta^{\alpha}{}_{\beta} := \frac{\delta L}{\delta \Gamma_{\alpha}{}^{\beta}} = \rho(L^{\alpha}{}_{\beta})\Psi \wedge \frac{\partial L}{\partial(D\Psi)} \qquad \text{hypermomentum} \tag{37} $$

Here $\rho(L^{\alpha}{}_{\beta})$ are infinitesimal operators of the Lie algebra of $GL(n, R)$ associated with Ψ.

The hypermomentum current Δ^{α}_{β} can be decomposed into its irreducible pieces comprising the spin, the dilation, and the shear currents:

$$\Delta_{\alpha\beta} = \tau_{\alpha\beta} + \frac{1}{n} g_{\alpha\beta}\Delta + \not{\Delta}_{\alpha\beta} \tag{38}$$

The shear current $\not{\Delta}_{\alpha\beta} := \Delta_{(\alpha\beta)} - (1/n)g_{\alpha\beta}\Delta$, which is symmetric and trace-free, and the dilation current $\Delta := \Delta^{\gamma}_{\gamma}$ both lead beyond the Poincaré gauge theory (PGT) and its (metric-compatible) Riemann-Cartan spacetime. Not surprisingly, the dilation current, an $(n-1)$-form, couples to the Weyl 1-form in accordance with (37):[12]

$$\Delta := \frac{\delta L}{\delta \Gamma^{\gamma}_{\gamma}} = \frac{2}{n}\frac{\delta L}{\delta Q} \tag{39}$$

The material Lagrangian n-form L is scalar-valued. Hence, it should be invariant under diffeomorphisms of spacetime and under local $GL(n, R)$ transformations. Provided the matter field equation

$$\frac{\delta L}{\delta \Psi} = 0 \tag{40}$$

is fulfilled, we find the first Noether identity

$$D\Sigma_{\alpha} = (e_{\alpha} \rfloor T^{\beta}) \wedge \Sigma_{\beta} + (e_{\alpha} \rfloor R_{\beta}{}^{\gamma}) \wedge \Delta^{\beta}{}_{\gamma} - \tfrac{1}{2}(e_{\alpha} \rfloor Q_{\beta\gamma})\sigma^{\beta\gamma} \tag{41}$$

and the second Noether identity

$$D\Delta^{\alpha}{}_{\beta} + \vartheta^{\alpha} \wedge \Sigma_{\beta} - g_{\beta\gamma}\sigma^{\alpha\gamma} = 0 \tag{42}$$

which are differential identities for energy-momentum and hypermomentum, respectively. Observe that (42) does not depend on the Abelian part of the connection, i.e., the Weyl 1-form Q. We apply (23) and find

$$^{\dagger}D\Delta^{\alpha}{}_{\beta} + \vartheta^{\alpha} \wedge \Sigma_{\beta} - \sigma^{\alpha}{}_{\beta} = 0 \tag{43}$$

We can project out the trace of (42) or (43),

$$d\Delta + \vartheta^{\gamma} \wedge \Sigma_{\gamma} - \sigma^{\gamma}_{\gamma} = 0 \tag{44}$$

for the dilation current Δ. Then the trace-free part is left over

$$^{\dagger}D\bar{\Delta}^{\alpha}{}_{\beta} + \left(\vartheta^{\alpha} \wedge \Sigma_{\beta} - \frac{1}{n}\delta^{\alpha}_{\beta}\vartheta^{\gamma} \wedge \Sigma_{\gamma}\right) - \left(\sigma^{\alpha}{}_{\beta} - \frac{1}{n}\delta^{\alpha}_{\beta}\sigma^{\gamma}_{\gamma}\right) = 0 \tag{45}$$

[12]This shows that the Weyl 1-form has nothing to do with the electromagnetic potential, as once surmised by Weyl, but rather with the well-established dilation current of canonical field theory. Incidentally, the variation in (39) with respect to Q is executed such that $g_{\alpha\beta}$, ϑ^{α}, and $^{\dagger}\Gamma_{\alpha}{}^{\beta}$ are kept constant.

for the trace-free intrinsic hypermomentum current

$$\bar{\Delta}^{\alpha}{}_{\beta} := \Delta^{\alpha}{}_{\beta} - \frac{1}{n}\,\delta^{\alpha}_{\beta}\Delta^{\gamma}_{\gamma} \tag{46}$$

We lower the index α in (45) and find

$$^{\dagger}D\bar{\Delta}_{\alpha\beta} + {}^{\dagger}Q_{\mu\alpha} \wedge \bar{\Delta}^{\mu}{}_{\beta} + \left(\vartheta_{\alpha} \wedge \Sigma_{\beta} - \frac{1}{n}\,g_{\alpha\beta}\vartheta^{\gamma} \wedge \Sigma_{\gamma}\right) - \left(\sigma_{\alpha\beta} - \frac{1}{n}\,g_{\alpha\beta}\sigma^{\gamma}_{\gamma}\right) = 0 \tag{47}$$

Now it is trivial to decompose (47) into its symmetric trace-free and its antisymmetric pieces. Since $\bar{\Delta}_{\alpha\beta} = \not{\Delta}_{\alpha\beta} + \tau_{\alpha\beta}$, we have

$$^{\dagger}D\not{\Delta}_{\alpha\beta} + {}^{\dagger}Q_{\mu(\alpha} \wedge \bar{\Delta}^{\mu}{}_{\beta)} + \left(\vartheta_{(\alpha} \wedge \Sigma_{\beta)} - \frac{1}{n}\,g_{\alpha\beta}\vartheta^{\gamma} \wedge \Sigma_{\gamma}\right) - \left(\sigma_{\alpha\beta} - \frac{1}{n}\,g_{\alpha\beta}\sigma^{\gamma}_{\gamma}\right) = 0 \tag{48}$$

and

$$^{\dagger}D\tau_{\alpha\beta} + {}^{\dagger}Q_{\mu[\alpha} \wedge \bar{\Delta}^{\mu}{}_{\beta]} + \vartheta_{[\alpha} \wedge \Sigma_{\beta]} = 0 \tag{49}$$

Accordingly, (44) represents the law for the dilation current and (48) that for the shear current, whereas (49) is the general version of the angular momentum law.

It was a big surprise to us that the Mindlin-type equilibrium condition (5) for continua with microstructure, which has a very transparent physical interpretation in 3-dimensional continuum mechanics, has this nice Noether analogue (42) in 4 or in arbitrary n dimensions. This is one of the reasons which convinces us that there is a future for metric-affine geometry in continuum physics as well as in extended theories of gravitation. Another reason is, of course, the Kröner argument (Kröner, 1990) of the relation of a distribution of point stacking faults to the nonmetricity of a "continuized" crystal.

4. COUNTING THE NOETHER AND BIANCHI IDENTITIES AND PREVIEW OF THE RESULTS

The task to be carried out is now obvious. We want to map the Noether identities (41), (42) into the Bianchi identities (19)-(21). Let us first of all repeat the results for the case of a Riemann-Cartan spacetime (Hehl and McCrea, 1986). These are summarized in Tables I and II and the sense in which Bianchi and Noether are mapped to one another is as explained in Section 1 and in Hehl and McCrea (1986).

Table I. Noethers and Bianchis in Four-dimensional Riemann-Cartan Spacetime[a]

2nd Noether	6	$\longleftrightarrow$	$\left.\begin{matrix}9\\6\\1\end{matrix}\right\}$	1st Bianchi
1st Noether	4	$\longleftrightarrow$	$\left.\begin{matrix}16\\4\\4\#\end{matrix}\right\}$	2nd Bianchi

[a]Here the Noether identities are irreducible under the action of the Lorentz group $SO(3,1)$. On the other hand, the Bianchi identities (8) and (9) can be further decomposed into irreducible pieces with dimensionality as shown. We see how nicely and almost uniquely these two mappings work; compare equations (6) and (7). Since the piece marked with # does not survive in 3 dimensions, it does not seem to be a likely candidate for the mapping of the Noether identities in 4 dimensions either.

5. THE MAPPING PRESCRIPTIONS

In Section 1 we have seen that the mappings in the Riemann-Cartan space are just represented by the field equations of the ECT, the Lagrangian of which is $(1/2\kappa)R_\alpha{}^\beta \wedge \eta_\beta{}^\alpha$. This suggests that we should take the analogous Lagrangian with action

$$W = \frac{1}{2\kappa}\int R_\alpha{}^\beta \wedge \eta^\alpha{}_\beta \tag{50}$$

for the metric-affine space and vary it with respect to the appropriate potentials $(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha{}^\beta)$. The cosmological term $\Lambda\eta$ and, in 4 dimensions, the parity-violating term $R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta$, as well as the related boundary

Table II. Noethers and Bianchis in Three-Dimensional Riemann-Cartan Stress Space[a]

2nd Noether	3	$\longleftrightarrow$	3	1st Bianchi
1st Noether	3	$\longleftrightarrow$	3	2nd Bianchi

[a]In 3 dimensions, both Noether and Bianchi identities are irreducible under $SO(3)$. Although the table seems to allow for the possibility of a mapping between 1st Noether and 1st Bianchi and similarly for the 2nd ones, this is, in fact, *not* possible with one constant of proportionality, due to the physical dimensions of the quantities involved. dim(stress) = force/length^2, dim(moment stress) = force/length, dim(torsion) = 1/length, dim(curvature) = 1/length^2.

term $d(\vartheta_\alpha \wedge T^\alpha)$, will be omitted here; see, however Hehl and McCrea (1986) and Mielke *et al.* (1989).

We need to be cautious. The action is invariant under *projective* transformations of the connection

$$\Gamma_\alpha{}^\beta \to \Gamma_\alpha{}^\beta - A\delta_\alpha^\beta \tag{51}$$

where A is a scalar-valued 1-form. Then the curvature transforms as follows [see (28)]:

$$R_\alpha{}^\beta \to R_\alpha{}^\beta - dA\,\delta_\alpha^\beta \tag{52}$$

Because of $\eta_{\alpha\beta} = -\eta_{\beta\alpha}$ in (50), the additional piece from (52) drops out. As a consequence, the field equations to be derived from (50) cannot control the trace Γ_γ^γ of the connection. This is why we introduced the volume-preserving connection in (23), which uniquely defines the class of all connections obtainable from it by projective transformations of the type (51). The Einstein–Hilbert-type action is then

$$^\dagger W = \frac{1}{2\kappa}\int {}^\dagger R_\alpha{}^\beta \wedge \eta_\beta{}^\alpha = \frac{1}{2\kappa}\int R_\alpha{}^\beta \wedge \eta_\beta{}^\alpha = W \tag{53}$$

In order to have field equations which will determine also the trace of the connection we shall have to modify (53) in such a way that projective invariance no longer holds. To achieve this, we add the seemingly simplest term necessary, namely one that is quadratic in the Weyl 1-form:

$$L_Q := \frac{f}{2\kappa} Q \wedge {}^*Q \tag{54}$$

The dimensionless constant f here will enable us to trace back the terms which arise because of this additional term in the Lagrangian.

Denoting the material Lagrangian by L, we vary the action

$$\int \left(\frac{1}{2\kappa} R_\alpha{}^\beta \wedge \eta_\beta{}^\alpha + L_Q + L\right) \tag{55}$$

with respect to the potentials $(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha^\beta)$ and find

$$\begin{aligned} 2\kappa\sigma^{\alpha\beta} &= -2\,{}^\dagger R^{(\alpha}{}_\gamma \wedge \eta^{\beta)\gamma} + g^{\alpha\beta}\,{}^\dagger R^{\gamma\delta} \wedge \eta_{\gamma\delta} \\ &\quad - f\{g^{\alpha\beta} D\,{}^*Q + Q \wedge [2\vartheta^{(\alpha} \wedge (e^{\beta)} \rfloor {}^*Q) - g^{\alpha\beta}\,{}^*Q]\} \end{aligned} \tag{M0}$$

$$2\kappa\Sigma_\alpha = {}^\dagger R^{\gamma\delta} \wedge \eta_{\gamma\delta\alpha} + f\{(e_\alpha \rfloor Q)^*Q + Q \wedge (e_\alpha \rfloor {}^*Q)\} \tag{M1}$$

$$2\kappa\Delta_{\alpha\beta} = {}^\dagger T^\gamma \wedge \eta_{\alpha\beta\gamma} + {}^\dagger Q_{\alpha\gamma} \wedge \eta^\gamma{}_\beta - f g_{\alpha\beta}\,{}^*Q \tag{M2}$$

The right-hand sides of these three equations, for the special case of $f = 0$, are projectively invariant and we have made this manifest by writing them in terms of the volume-preserving connection of (23); see also (28)–(30).

For $Q_{\alpha\beta}=0$ and $\Delta_{(\alpha\beta)}=0$, (M1) and (M2) reduce to (6) and (7), respectively, since in a Riemann-Cartan space ${}^{\dagger}\Gamma_{\alpha}{}^{\beta}=\Gamma_{\alpha}{}^{\beta}$. In the general case ($Q_{\alpha\beta}\neq 0$), the trace-free symmetric part of (M2) coincides with (13)—and this is the novel core formula relating the *shear current* $\not\Delta_{\alpha\beta}$ of (13) to the *trace-free nonmetricity* ${}^{\dagger}Q_{\alpha\beta}$:

$$\not\Delta_{\alpha\beta}=\frac{1}{2\kappa}\,{}^{\dagger}Q_{\gamma(\alpha}\wedge\eta_{\beta)}{}^{\gamma} \tag{56}$$

Incidentally, for $f=0$, the prescriptions (M0)-(M2) map the canonical currents, which are Hodge dual to 1-forms, via the right dual $\frac{1}{2}\eta_{\alpha_1\cdots\alpha_n}$ into the field strengths appropriately amended by wedge products with the coframe.

For future use we shall need the traces of the mapping prescriptions. A short calculation yields

$$\kappa\sigma_{\gamma}^{\gamma}={}^{\dagger}R^{\gamma\delta}\wedge\eta_{\gamma\delta}-f(2d\,{}^{*}Q+Q\wedge{}^{*}Q) \tag{57}$$

$$\kappa\vartheta^{\gamma}\wedge\Sigma_{\gamma}={}^{\dagger}R^{\gamma\delta}\wedge\eta_{\gamma\delta}-fQ\wedge{}^{*}Q \tag{58}$$

$$\kappa\Delta_{\gamma}^{\gamma}=-2f\,{}^{*}Q \tag{59}$$

One can see at a glance that these three equations satisfy the dilation identity (14),

$$d\Delta_{\gamma}^{\gamma}+\vartheta^{\gamma}\wedge\Sigma_{\gamma}-\sigma_{\gamma}^{\gamma}=0 \tag{60}$$

as they should. An arbitrary dilation current is only allowed if $f\neq 0$. Therefore the supplementary Lagrangian (54) is required in order *not* to constrain the hypermomentum current in (M2) *a priori.*

6. THE IRREDUCIBLE DECOMPOSITIONS OF THE BIANCHI IDENTITIES

The description of how the Noether identities are mapped to the Bianchi identities via the field equations (M0)-(M2) will be greatly facilitated by a knowledge of the irreducible parts of the 0th, 1st, and 2nd Bianchi identities.

6.1. Decomposition of the Zeroth Bianchi Identity

The 0th Bianchi identity is given by

$$B_{\alpha\beta}\equiv 0 \tag{61}$$

with

$$B_{\alpha\beta}=DQ_{\alpha\beta}-2R_{(\alpha\beta)} \tag{62}$$

This is a symmetric tensor-valued 2-form and in *4-dimensional* metric-affine spacetime it may be decomposed into a sum of irreducible pieces as follows.

Let

$$\Omega_\alpha = e^\beta \rfloor B_{\alpha\beta}, \qquad \Xi_\alpha = {}^*(B_{\alpha\beta} \wedge \vartheta^\beta), \qquad B = B_\gamma{}^\gamma \tag{63}$$

and

$$\Phi_\alpha = \Omega_\alpha + \tfrac{1}{2} e_\alpha \rfloor (\Omega_\mu \wedge \vartheta^\mu), \qquad \Psi_\alpha = \Xi_\alpha + \tfrac{1}{2} e_\alpha \rfloor (\Xi_\mu \wedge \vartheta^\mu) \tag{64}$$

Note that Φ_α and Ψ_α satisfy

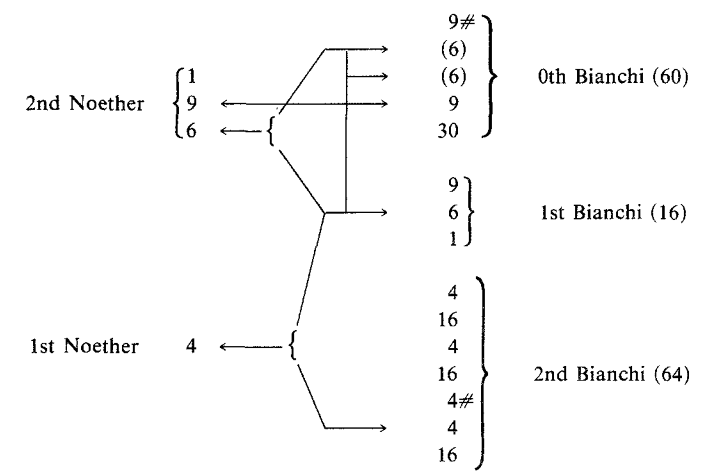
$$e_\alpha \rfloor \Phi^\alpha = 0, \qquad \Phi_\alpha \wedge \vartheta^\alpha = 0 \tag{65}$$

$$e_\alpha \rfloor \Psi^\alpha = 0, \qquad \Psi_\alpha \wedge \vartheta^\alpha = 0 \tag{66}$$

The irreducible decomposition may then be written as (compare Table III)

$$\begin{gathered} B_{\alpha\beta} = {}^{(1)}B_{\alpha\beta} + {}^{(2)}B_{\alpha\beta} + {}^{(3)}B_{\alpha\beta} + {}^{(4)}B_{\alpha\beta} + {}^{(5)}B_{\alpha\beta} \\ 60 = 9 + 6 + 6 + 9 + 30 \end{gathered} \tag{67}$$

Table III. Noethers and Bianchis in Four-Dimensional Metric-Affine Spacetime[a]



[a]The 2nd Noether identity for hypermomentum is given in (42). It splits into 3 pieces, its trace (44), its trace-free symmetric part (48), and its antisymmetric part (49). The 1st Noether identity is given in (41) and it represents the momentum law. The Bianchi identities are to be found in (19)-(21). The arrows represent the mappings derived in Section 7. The first arrow for 2nd Noether (9) corresponds to (116), the second one for 2nd Noether (6) to (118) with (120) and (123), and the last one for 1st Noether to (124) with (125) and (126). The parentheses around the two 6-dimensional subspaces in the 0th Bianchi indicate the noncanonical character of this splitting. The pieces marked with # do not survive in 3 dimensions and are therefore uninteresting from a physical point of view. The trace (44) of the 2nd Noether identity has no Bianchi image, except for a mapping to $ddQ = 0$.

where

$$^{(1)}B_{\alpha\beta} = -\tfrac{1}{2}\,{}^*\{\vartheta_{(\alpha} \wedge \Psi_{\beta)}\} \tag{68}$$

$$^{(2)}B_{\alpha\beta} = \tfrac{1}{6}\,{}^*[\vartheta_{(\alpha} \wedge \{e_{\beta)} \rfloor (\Xi_\mu \wedge \vartheta^\mu)\}] \tag{69}$$

$$^{(3)}B_{\alpha\beta} = -\tfrac{1}{6}\vartheta_{(\alpha} \wedge [e_{\beta)} \rfloor (\Omega_\mu \wedge \vartheta^\mu)] + \tfrac{1}{6}g_{\alpha\beta}B \tag{70}$$

$$^{(4)}B_{\alpha\beta} = \tfrac{1}{2}\vartheta_{(\alpha} \wedge \Phi_{\beta)} \tag{71}$$

$$^{(5)}B_{\alpha\beta} = B_{\alpha\beta} - {}^{(1)}B_{\alpha\beta} - {}^{(2)}B_{\alpha\beta} - {}^{(3)}B_{\alpha\beta} - {}^{(4)}B_{\alpha\beta} \tag{72}$$

The invariant subspaces $^{(1)}B$, $^{(4)}B$, and $^{(5)}B$ are canonical, whereas the sum of the two isomorphic subspaces $^{(2)}B$ and $^{(3)}B$ may be split into two invariant subspaces in an infinite number of ways, of which (69), (70) represent one possibility. This is perhaps more transparent in terms of the tensor components $B_{\alpha\beta\gamma\delta}$ of $B_{\gamma\delta}$ given by

$$B_{\gamma\delta} = \tfrac{1}{2}B_{\alpha\beta\gamma\delta}\vartheta^\alpha \wedge \vartheta^\beta \tag{73}$$

Let

$$F_{\alpha\beta} = B_{\gamma[\alpha\beta]}{}^\gamma \qquad \text{and} \qquad H_{\alpha\beta} = B_{\alpha\beta\gamma}{}^\gamma \tag{74}$$

Then

$$^{(2)}B_{\gamma\delta} = \tfrac{1}{2}\,{}^{(2)}B_{\alpha\beta\gamma\delta}\vartheta^\alpha \wedge \vartheta^\beta \qquad \text{and} \qquad {}^{(3)}B_{\gamma\delta} = \tfrac{1}{2}\,{}^{(3)}B_{\alpha\beta\gamma\delta}\vartheta^\alpha \wedge \vartheta^\beta \tag{75}$$

where

$$^{(2)}B_{\alpha\beta\gamma\delta} = \tfrac{1}{6}(-4g_{[\alpha(\gamma}F_{\delta)\beta]} + 2g_{\gamma\delta}F_{\alpha\beta} - 2g_{[\alpha(\gamma}H_{\delta)\beta]} + g_{\gamma\delta}H_{\alpha\beta}) \tag{76}$$

and

$$^{(3)}B_{\alpha\beta\gamma\delta} = \tfrac{1}{6}(-4g_{[\alpha(\gamma}F_{\delta)\beta]} - g_{\gamma\delta}H_{\alpha\beta}) \tag{77}$$

Thus, although, from the point of view of the tensor components, a splitting into the two 6-dimensional, isomorphic invariant subspaces represented by $F_{\alpha\beta}$ and $H_{\alpha\beta}$ would seem to be most natural, combinations of the type (76) and (77) are equally valid as invariant subspaces under the Lorentz group $SO(1,3)$.

For future reference we note that, in terms of tensor components, the 1-form Φ_α that occurs in $^{(4)}B_{\alpha\beta}$ is given by

$$\Phi_\alpha = B_{\mu(\alpha\beta)}{}^\mu\vartheta^\beta \tag{78}$$

$B_{\mu(\alpha\beta)}{}^\mu$ is trace-free as well as symmetric, as is also evident from (65).

In *3 dimensions* Ξ_α is a 0-form and

$$\Psi_\alpha = \Xi_\alpha - e_\alpha \rfloor (\Xi_\mu \vartheta^\mu) = 0$$

Hence ${}^{(1)}B_{\alpha\beta}$ drops out and we find the decomposition (see also Table IV)

$$\begin{aligned} B_{\alpha\beta} &= {}^{(2)}B_{\alpha\beta} + {}^{(3)}B_{\alpha\beta} + {}^{(4)}B_{\alpha\beta} + {}^{(5)}B_{\alpha\beta} \\ 18 &= 3+3+5+7 \end{aligned} \tag{79}$$

where

$${}^{(2)}B_{\alpha\beta} = \tfrac{2}{5}\,{}^{*}[\Xi_{(\alpha}\vartheta_{\beta)}] \tag{80}$$

$${}^{(3)}B_{\alpha\beta} = -\tfrac{1}{5}\vartheta_{(\alpha} \wedge [e_{\beta)} \rfloor (\Omega_{\mu} \wedge \vartheta^{\mu})] + \tfrac{1}{5} g_{\alpha\beta} B \tag{81}$$

$${}^{(4)}B_{\alpha\beta} = \tfrac{2}{3}\vartheta_{(\alpha} \wedge \Phi_{\beta)} \tag{82}$$

$${}^{(5)}B_{\alpha\beta} = B_{\alpha\beta} - {}^{(2)}B_{\alpha\beta} - {}^{(3)}B_{\alpha\beta} - {}^{(4)}B_{\alpha\beta} \tag{83}$$

6.2. Decomposition of the First Bianchi Identity

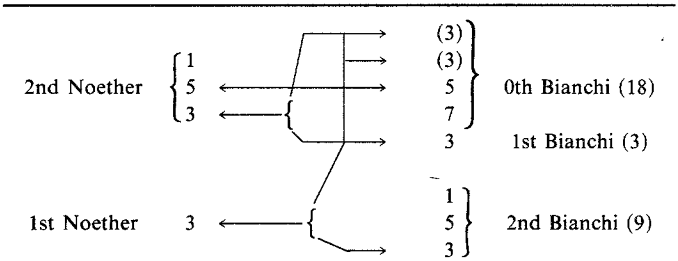
Let

$$B'^{\alpha} = DT^{\alpha} - R_{\beta}{}^{\alpha} \wedge \vartheta^{\beta} \tag{84}$$

so that the 1st Bianchi identity is

$$B'^{\alpha} \equiv 0 \tag{85}$$

Table IV. Noethers and Bianchis in Three-Dimensional Metric-Affine Stress Space[a]



[a]This table is applicable to 3-dimensional continuum statics where 1st Noether represents the balance of force and 2nd Noether the balance of hyperforce with and without moment. The decomposition of 0th Bianchi can be found in (79), 1st Bianchi is irreducible, and for 2nd Bianchi compare the last paragraph of Section 6. Here we propose the new mapping between the trace-free symmetric part of 2nd Noether (5 components) and an irreducible piece of 0th Bianchi. This is basically achieved by means of our new key formula (56): $\not{\Delta}_{\alpha\beta} = (1/2\kappa)\,{}^{\dagger}Q_{\gamma(\alpha} \wedge \eta_{\beta)}{}^{\gamma}$. Thereby the intrinsic shear hyperstress $\not{\Delta}_{\alpha\beta}$ is related to the trace-free part of the nonmetricity $Q_{\gamma\delta}$ of the stress space. The trace of the 2nd Noether identity has no Bianchi image, except for a mapping to $ddQ = 0$.

B'^{α} is a vector-valued 3-form and, for a 4-dimensional spacetime, its irreducible decomposition is given by (see Hehl and McCrea, 1986)

$$\begin{aligned} B'^{\alpha} &= {}^{(1)}B'^{\alpha} + {}^{(2)}B'^{\alpha} + {}^{(3)}B'^{\alpha} \\ 16 &= 9+6+1 \end{aligned} \tag{86}$$

with

$${}^{(2)}B'^{\alpha} = \tfrac{1}{2}(e_{\beta} \rfloor B'^{\beta}) \wedge \vartheta^{\alpha}, \qquad {}^{(3)}B'^{\alpha} = \tfrac{1}{4}\, {}^{*}(B'^{\beta} \wedge \vartheta_{\beta})\eta^{\alpha} \tag{87}$$

and

$${}^{(1)}B'^{\alpha} = B'^{\alpha} - {}^{(2)}B'^{\alpha} - {}^{(3)}B'^{\alpha} \tag{88}$$

In terms of tensor components $B'^{\alpha\beta}$, where $B'^{\alpha} = B'^{\alpha\beta}\eta_{\beta}$, ${}^{(1)}B'^{\alpha\beta}$ is the trace-free symmetric, ${}^{(2)}B'^{\alpha\beta}$ the antisymmetric, and ${}^{(3)}B'^{\alpha\beta}$ the trace part. In 3 dimensions, a vector-valued 3-form is obviously irreducible as it stands.

6.3. Decomposition of the Second Bianchi Identity

The left-hand side of the 2nd Bianchi identity

$$B''^{\alpha\beta} \equiv 0 \tag{89}$$

where

$$B''^{\ \beta}_{\alpha} = DR_{\alpha}{}^{\beta} \tag{90}$$

is a 2nd-order, tensor-valued 3-form. As a first step in deriving its irreducible decomposition, we split it into its antisymmetric and symmetric parts,

$$\begin{aligned} B''^{\alpha\beta} &= A^{\alpha\beta} + S^{\alpha\beta}, \qquad A^{\alpha\beta} = B''^{[\alpha\beta]}, \qquad S^{\alpha\beta} = B''^{(\alpha\beta)} \\ 64 &= 24+40 \end{aligned} \tag{91}$$

The irreducible pieces of $A^{\alpha\beta}$ have already been derived in Hehl and McCrea (1986):

$$\begin{aligned} A^{\alpha\beta} &= {}^{(1)}A^{\alpha\beta} + {}^{(2)}A^{\alpha\beta} + {}^{(3)}A^{\alpha\beta} \\ 24 &= 16+4+4 \end{aligned} \tag{92}$$

with

$${}^{(2)}A^{\alpha\beta} = \tfrac{2}{3} Y^{[\alpha}\eta^{\beta]}, \qquad Y^{\alpha} = {}^{*}(A^{\alpha\beta} \wedge \vartheta_{\beta}) \tag{93}$$

$${}^{(3)}A^{\alpha\beta} = \tfrac{1}{6} W \wedge \vartheta^{\alpha} \wedge \vartheta^{\beta}, \qquad W = e_{\mu} \rfloor e_{\nu} \rfloor A^{\nu\mu} \tag{94}$$

$${}^{(1)}A^{\alpha\beta} = A^{\alpha\beta} - {}^{(2)}A^{\alpha\beta} - {}^{(3)}A^{\alpha\beta} \tag{95}$$

Note the properties

$$(A^{\alpha\beta} - {}^{(2)}A^{\alpha\beta}) \wedge \vartheta_{\beta} = 0 \tag{96}$$

$$e_{\alpha} \rfloor e_{\beta} \rfloor (A^{\alpha\beta} - {}^{(3)}A^{\alpha\beta}) = 0 \tag{97}$$

$${}^{(1)}A^{\alpha\beta} \wedge \vartheta_{\beta} = 0, \qquad e_{\alpha} \rfloor e_{\beta} \rfloor {}^{(1)}A^{\alpha\beta} = 0 \tag{98}$$

To find the irreducible pieces of the symmetric part $S^{\alpha\beta}$, we proceed as follows: Let

$$P^{\alpha} = e_{\beta} \rfloor S^{\alpha\beta} \qquad (\Rightarrow e_{\alpha} \rfloor P^{\alpha} = 0) \tag{99}$$

and let

$$V^{\alpha\beta} = \tfrac{2}{3} P^{(\alpha} \wedge \vartheta^{\beta)}, \qquad Z^{\alpha\beta} = S^{\alpha\beta} - V^{\alpha\beta} \tag{100}$$

Then

$$e_{\beta} \rfloor Z^{\alpha\beta} = 0 \tag{101}$$

$Z^{\alpha\beta}$ and $V^{\alpha\beta}$ are the irreducible parts of $S^{\alpha\beta}$ under $GL(4, R)$. If $S^{\alpha\beta\gamma}$ are the components of $S^{\alpha\beta}$ in accordance with $S^{\alpha\beta} = S^{\alpha\beta\gamma}\eta_{\gamma}$, it follows from (100) and (101) that the corresponding components of $Z^{\alpha\beta}$ are the completely symmetric $S^{(\alpha\beta\gamma)}$. To get the irreducible pieces under the Lorentz group, we take out the traces of $Z^{\alpha\beta}$ and $V^{\alpha\beta}$. For $Z^{\alpha\beta}$ this yields

$$\begin{aligned} Z^{\alpha\beta} &= {}^{(1)}S^{\alpha\beta} + {}^{(2)}S^{\alpha\beta} \\ 20 &= 16 + 4 \end{aligned} \tag{102}$$

where, defining $Z := Z^{\gamma}_{\gamma}$ and $Z^{\alpha} := Z \wedge \vartheta^{\alpha}$,

$${}^{(2)}S^{\alpha\beta} = \tfrac{1}{6} Z g^{\alpha\beta} - \tfrac{1}{3}(e^{(\beta} \rfloor Z^{\alpha)}) \tag{103}$$

and

$${}^{(1)}S^{\alpha\beta} = Z^{\alpha\beta} - {}^{(2)}S^{\alpha\beta} \tag{104}$$

The term ${}^{(1)}S^{\alpha\beta}$ has the trace-free properties

$${}^{(1)}S_{\alpha\beta} \wedge \vartheta^{\beta} = 0, \qquad {}^{(1)}S^{\alpha}_{\alpha} = 0 \tag{105}$$

For $V^{\alpha\beta}$ we get

$$\begin{aligned} V^{\alpha\beta} &= {}^{(3)}S^{\alpha\beta} + {}^{(4)}S^{\alpha\beta} \\ 20 &= 16 + 4 \end{aligned} \tag{106}$$

where, if $V = P^{\gamma} \wedge \vartheta_{\gamma}$ and $V^{\alpha} = V \wedge \vartheta^{\alpha}$,

$${}^{(4)}S^{\alpha\beta} = \tfrac{2}{9}\{ V g_{\alpha\beta} + (e_{(\beta} \rfloor V_{\alpha)})\} \tag{107}$$

and

$${}^{(3)}S^{\alpha\beta} = V^{\alpha\beta} - {}^{(4)}S^{\alpha\beta} \tag{108}$$

Note that

$${}^{(3)}S_{\alpha\beta} \wedge \vartheta^{\beta} = 0 \qquad \text{and} \qquad {}^{(3)}S^{\alpha}_{\alpha} = 0 \tag{109}$$

To sum up, the left-hand side of the 2nd Bianchi identity in a 4-dimensional metric-affine spacetime may be written in terms of its seven irreducible parts as follows:

$$B''^{\alpha\beta} = {}^{(1)}A^{\alpha\beta} + {}^{(2)}A^{\alpha\beta} + {}^{(3)}A^{\alpha\beta} + {}^{(1)}S^{\alpha\beta} + {}^{(2)}S^{\alpha\beta} + {}^{(3)}S^{\alpha\beta} + {}^{(4)}S^{\alpha\beta}$$
$$64 = 16 + 4 + 4 + 16 + 4 + 16 + 4 \tag{110}$$

Here ${}^{(n)}A^{\alpha\beta}$ $(n = 1, 2, 3)$ are defined by (93)-(95), while ${}^{(m)}S^{\alpha\beta}$ $(m = 1, \ldots, 4)$ are as given by (103), (104), (107), and (108).

In three dimensions, the irreducible decomposition of the (2nd-order) tensor-valued 3-forms consists simply of the trace, the trace-free symmetric, and the antisymmetric parts (see Table IV).

7. THE NOETHER–BIANCHI MAPPINGS

The right-hand sides of (M0)-(M2) are the variational derivatives

$$-2(\delta V/\delta g_{\alpha\beta}), \qquad -(\delta V/\delta\vartheta^{\alpha}), \qquad -g_{\alpha\gamma}(\delta V/\delta\Gamma_{\gamma}^{\beta})$$

respectively of

$$V := \frac{1}{2\kappa} R_{\alpha}{}^{\beta} \wedge \eta_{\beta}{}^{\delta} + L_Q$$

Hence, when (M0)-(M2) are substituted into the different irreducible pieces of the Noether identities (41) and (42), they yield true, and not simply "weak" or "on-shell," identities, namely the Noether identities for V. To investigate the correspondence between *these* Noether identities and the Bianchi identities, we make the substitutions

$$DQ_{\alpha\beta} \to B_{\alpha\beta} + 2R_{(\alpha\beta)}, \qquad DT^{\alpha} \to B'^{\alpha} + R_{\beta}{}^{\alpha} \wedge \vartheta^{B}, \qquad DR_{\alpha}{}^{\beta} \to B''^{\beta}_{\alpha} \tag{111}$$

in the explicit expressions for the irreducible parts of the Noether identities for V that we have obtained. When we have done this, we find that everything cancels out except for terms involving irreducible parts of the $B^{\alpha\beta}$, B'^{α}, and $B''^{\alpha\beta}$, i.e., of the three Bianchi identities. It is in this sense that we can map the different irreducible pieces of Noether into corresponding irreducible pieces of Bianchi.

Let us begin with the *trace-free symmetric part of 2nd Noether*, which, according to (48), may be written for 4-dimensional spacetime in the form

$$D\not{\Delta}_{\alpha\beta} + Q_{\mu(\alpha} \wedge \bar{\Delta}^{\mu}{}_{\beta)} + \vartheta_{(\alpha} \wedge \Sigma_{\beta)} - \tfrac{1}{4}g_{\alpha\beta}\vartheta^{\gamma} \wedge \Sigma_{\gamma} - (\sigma_{\alpha\beta} - \tfrac{1}{4}g_{\alpha\beta}\sigma_{\gamma}^{\gamma}) = 0 \tag{112}$$

where $\not{\Delta}_{\alpha\beta}$ is the shear current defined by (13), while

$$\bar{\Delta}^{\alpha}{}_{\beta} = \Delta^{\alpha}{}_{\beta} - \tfrac{1}{4}\delta^{\alpha}_{\beta}\Delta^{\gamma}_{\gamma} = \not{\Delta}^{\alpha}{}_{\beta} + \tau^{\alpha}{}_{\beta} \tag{113}$$

By substituting (M0)-(M2) into (112) and using (111), we find that we are left with

$$B_{\mu(\alpha} \wedge \eta^{\mu}{}_{\beta)} = 0 \tag{114}$$

where $B_{\alpha\beta}$ is the left-hand side of the 0th Bianchi identity as in (62). By noting that

$$B_{\mu(\alpha} \wedge \eta^{\mu}{}_{\beta)} = B_{\mu(\alpha\beta)}{}^{\mu}\eta \tag{115}$$

and comparing with (78) and (71), we see that (M0)-(M2) maps the trace-free symmetric part of 2nd Noether into the irreducible piece

$${}^{(4)}B^{\alpha\beta} = 0 \tag{116}$$

of the 0th Bianchi identity, in the sense that we have explained above. The correspondence between the two 9-pieces of Table III has therefore been established.

The trace part of 2nd Noether (44) has only one component. It is not possible to map it to a suitable Bianchi identity by means of our mapping relations, except for the trivial mapping to $ddQ = 0$.

We now turn to the *antisymmetric part of the 2nd Noether* identity (49), which reads

$$D\tau_{\alpha\beta} + Q_{\mu[\alpha} \wedge \bar{\Delta}^{\mu}{}_{\beta]} + \vartheta_{[\alpha} \wedge \Sigma_{\beta]} = 0 \tag{117}$$

Substitution of (M1) and (M2) into (117) together with the substitutions (111) yields

$$B'^{\gamma} \wedge \eta_{\alpha\beta\gamma} + C_{\gamma[\alpha} \wedge \eta^{\gamma}{}_{\beta]} = 0 \tag{118}$$

with

$$C_{\alpha\beta} = B_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} B^{\gamma}_{\gamma} \tag{119}$$

For the first term on the left-hand side of (118) we have

$$B'^{\gamma} \wedge \eta_{\alpha\beta\gamma} = (e_{\gamma} \rfloor B'^{\gamma}) \wedge \eta_{\alpha\beta} \tag{120}$$

and therefore, by $(87)_1$, this term corresponds to the irreducible part ${}^{(2)}B'^{\alpha}$ of the 1st Bianchi identity. Hence, in the U_4 limit, the second term on the left-hand side of (118) drops out and we recover the map: antisymmetric 2nd Noether (6) $\leftrightarrow$ the irreducible piece (6) ${}^{(2)}B'^{\alpha}$ of the 1st Bianchi identity. However, here in the metric-affine case, we have an additional term $C_{\gamma[\alpha} \wedge \eta^{\gamma}_{\beta]}$ which in terms of frame components may be written

$$C_{\gamma[\alpha} \wedge \eta^{\gamma}{}_{\beta]} = -(\tfrac{1}{2} B_{\alpha\beta\gamma}{}^{\gamma} + B_{\gamma[\alpha\beta]}{}^{\gamma})\eta \tag{121}$$

Going back ot Ξ_{α} defined in (63), it is a straightforward matter to verify that

$$\Xi_{\mu} \wedge \vartheta^{\mu} = -(\tfrac{1}{2} B_{\alpha\beta\gamma}{}^{\gamma} + B_{\gamma[\alpha\beta]}{}^{\gamma})\eta^{\alpha\beta} \tag{122}$$

and hence, by using (69), that

$$^{(2)}B_{\alpha\beta}=0 \Leftrightarrow C_{\gamma[\alpha}\wedge\eta^{\gamma}{}_{\beta]}=0 \tag{123}$$

Thus, the additional term $C_{\gamma[\alpha}\wedge\eta^{\gamma}{}_{\beta]}$ in (122) corresponds to the 6-dimensional irreducible piece $^{(2)}B_{\alpha\beta}$ of the 0th Bianchi identity.

Finally, if we substitute (M0)-(M2) into the *1st Noether* identity (41) and use (111), we obtain

$$\tfrac{1}{2}B''^{[\gamma\delta]}\wedge\eta_{\gamma\delta\alpha}-f(e_\alpha\rfloor B)\wedge{}^*Q=0 \tag{124}$$

where $B''^{\alpha\beta}$ is as in (90) and

$$B=dQ-\tfrac{1}{2}R_\gamma^{\ \gamma}=\tfrac{1}{2}B_{\alpha\beta\gamma}{}^{\gamma}\vartheta^\alpha\wedge\vartheta^\beta \tag{125}$$

Since

$$B''^{[\gamma\delta]}\wedge\eta_{\gamma\delta\alpha}=-(e_\gamma\rfloor e_\delta\rfloor B''^{\gamma\delta})\wedge\eta_\alpha=W\wedge\eta_\alpha \tag{126}$$

it is clear from (94) that this term corresponds to the irreducible piece $^{(3)}A^{\alpha\beta}$ of the 2nd Bianchi identity. Furthermore, B of (125) corresponds to the trace of the 0th Bianchi identity, i.e., the 6-dimensional irreducible piece of the latter as defined in $(74)_2$. Thus, we see that, if $Q=0$, the 1st Noether identity is mapped into the 4-dimensional irreducible piece $^{(3)}A_{\alpha\beta}$ of the 2nd Bianchi identity, just as in the U_4 case of Hehl and McCrea (1986). However, for $Q\neq 0$, we have an additional term corresponding to an irreducible part of 0th Bianchi.

8. CONCLUSION

What is the net result of our work? In *relativistic field theory* we have shown that the Lagrangian (55) mediates between the material momentum and hypermomentum currents and the geometrical objects of a metric-affine spacetime in a reasonable and intuitively pleasing way. In particular, the Noether–Bianchi mappings give additional insight into the inner working of such theories. In *continuum mechanics* we have established that the stress space can be enriched by a trace-free nonmetricity which corresponds to the shear hyperstress as is, for instance, induced by point stacking faults in "continuized" crystals.

In both domains of application we recognized the rather special role played by the Weyl 1-form Q and the dilation current (hyperstress) Δ. In the Bianchi identities Q drops out altogether, as was shown in (31)-(33). The same is true for the 2nd Noether identity (45). The Δ only features in the trace part of the 2nd Noether identity (46), it drops out in the trace-free piece (47). This is due to the direct product structure of the linear group.[13]

[13] $GL(n,R)=(SL(n,R)\otimes P)\otimes R^+$.

It is then small wonder that we were unable to find a nontrivial Noether-Bianchi mapping for Δ.

From a physical point of view, we do not take the Lagrangian (55) too seriously, since it encompasses presumably unphysical contact interactions. We hope, however, that we were able to show that the concept of non-metricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, as unintuitive as it may appear at first, can make perfectly good sense in field theory and continuum mechanics.

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